Self-organized criticality in a hierarchical model of defects development

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We suggest a hierarchical model of defects development demonstrating critical behavior in a wide range of parameters that is naturally called the self-organized criticality. The kinetic equation is used for description of temporal evolution of the system. Conditions of appearance and healing of defects wholly govern the system behavior. Properties of the stationary solution are investigated. The model is found to show two kinds of behavior: stability and the self-organized criticality. The first one corresponds to small deformations in destruction experiments, when samples contain only the cracks of a few small ranges and there are no large cracks. The second one represents scaling properties of the world seismicity. The slope of magnitude-frequency dependence in the region of self-organized criticality is equal to unity for arbitrary parameters of the model. It is similar to the slope of the Gutenberg-Richter law defined for the world seismicity.

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I. INTRODUCTION

Complex systems consisting of objects with a large enough range of scales are often described by a magnitude-frequency relation. A power form of the relation is observed in various fields of science. Such a kind of relation describes, for example, a distribution of cities by the population and value of the population income. In seismology the magnitude-frequency relation is known as the Gutenberg-Richter law. It determines the relation between number of events and the energy of earthquakes E as follows:

$$\log_{10} N = a - b \log_{10} E \ . \qquad (1)$$

The power form for the magnitude-frequency law was thought to be unambiguous evidence of a critical point. However, this explanation is not suitable in many cases. Thus it is difficult to assume that various regions of the Earth with different conditions of evolution reach the critical point in the same time. Recently a power form of the magnitude-frequency relation was widely explained by the self-similar properties of structure and dynamics of the lithosphere [1]. The b value in the Gutenberg-Richter law is usually suggested as a reflection of fractal properties of local seismicity [2]. Another approach to the nature of power relations in complex systems was proposed by Bak *et al.* [3]. A model of avalanches exhibiting a power form of magnitude-frequency relation without any connection with a critical point was suggested. This kind of behavior was called the self-organized criticality. Recently systems demonstrating the self-organized criticality have drawn close attention in connection with seismicity modeling [4]; see also [5] for a review.

The lithosphere has been often considered as an hierarchical system of blocks [6–8]. An approach to the description of seismicity in terms of hierarchical systems is suggested in Refs. [8–11]. In the hierarchical systems considered earlier [8–10] a transition from stability to catastrophe was observed. We propose below a hierarchical model exhibiting a stability–self-organized-criticality transition.

Section II contains a general description of the system and the kinetic equations determining its dynamics. The properties of the stationary solution for different values of parameters are investigated in Secs. III–V. In Sec. VI the magnitude-frequency law is studied for different parameters.

II. GENERAL DESCRIPTION OF THE MODEL

The hierarchical system under consideration represents a tree with the branch number equal to 3 (Fig. 1). The state of each element of $(l+1)$th level is determined by the state of the corresponding group of three elements of the lth level. There are two possible states for each element of the system: the whole and the defect ones. Different configurations with i defects and $3-i$ whole

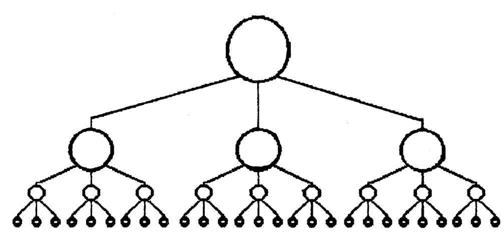

FIG. 1. Hierarchical system of defects. Each group of three elements of the lower level composes an element of the upper level.

elements in each group form possible states of the group. Configurations with 2 or 3 defects in one group are referred to below as the critical configurations. On the first level defects appear with constant intensity α_0. When a critical configuration appears on the lth level it causes the appearance of a defect of the $(l+1)$th level. The critical configuration appearing in the time moment t is healed in the time moment $(t+1)$. The other defects of lth level are healing with the healing intensity β_l. Thus there are two healing processes in the system: the determined healing of two or three defects in a group of elements and the stochastic healing of single defects.

Each element of the system can be in two possible states, so for a group of three elements there are eight different states. All configurations with the same number of defects have equal probabilities. The system is described by the densities $P^i(l,t), i=0,1,2,3$ with $P^i(l,t)$ being the density of groups in the lth level including i defects in time moment t. The appearing intensity of defects $\alpha(l,t)$ is defined for each time moment t and number of level l. It depends on the probability of there appearing a critical configuration on the previous level:

$$\alpha(l+1,t) = P^0(l,t)[3\alpha^2(l,t) - 2\alpha^3(l,t)] + 3P^1(l,t)[2\alpha(l,t) - \alpha^2(l,t)] \ . \qquad (2)$$

The appearing intensity of the initial level is constant:

$$\alpha(0,t) = \alpha_0 \ . \qquad (3)$$

The healing intensity of the single defects $\beta(l)$ is assumed to be constant in time and depends on the level as follows:

$$\beta(l) = \beta_0 c^l, \quad 0 < c \le 1 \ . \qquad (4)$$

The density of defect elements on the lth level $p(l,t)$ is expressed from densities of different configurations as follows:

$$p(l,t) = P^1(l,t) + 2P^2(l,t) + P^3(l,t) \ . \qquad (5)$$

The total number of defects in time moment $(t+1)$ is expressed as a sum of the number of defects which existed at time moment t and were not healing and the number of new defects appearing in time moment $(t+1)$. Hence the temporal evolution of density of defects in the lth level is expressed as follows:

$$p(l,t+1) = P^1(l,t)[1-\beta(l)] + [1-p(l,t)]\alpha(l,t) \ . \quad (6)$$

Similarly it is possible to define the temporal evolution of densities of different configurations for groups of lth level:

$$\begin{aligned} P^0(l,t+1) = {} & P^0(l,t)[1-\alpha(l,t)]^3 \\ & +3P^1(l,t)\beta(l)[1-\alpha(l,t)]^2 \\ & +3P^2(l,t)[1-\alpha(l,t)] + P^3(l,t) \ , \end{aligned} \qquad (7)$$

$$\begin{aligned} P^1(l,t+1) = {} & P^0(l,t)\alpha(l,t)[1-\alpha(l,t)]^2 \\ & +P^1(l,t)\{[1-\beta(l)][1-\alpha(l,t)]^2 \\ & +2\beta(l)\alpha(l,t)[1-\alpha(l,t)]\} \\ & +P^2(l,t)\alpha(l,t) \ , \end{aligned} \qquad (8)$$

$$\begin{aligned} P^2(l,t+1) = {} & P^0(l,t)\alpha^2(l,t)[1-\alpha(l,t)] \\ & +P^1(l,t)\{\beta(l)\alpha^2(l,t) \\ & +2[1-\beta(l)]\alpha(l,t)[1-\alpha(l,t)]\} \ , \end{aligned} \qquad (9)$$

$$\begin{aligned} P^3(l,t+1) = {} & P^0(l,t)\alpha^3(l,t) \\ & +3P^1(l,t)\alpha^2(l,t)[1-\beta(l)] \ . \end{aligned} \qquad (10)$$

These kinetic equations enable us to investigate the behavior of the system for different parameters α_0, β_0, c.

For $t \to \infty$ the densities of defects $P^i(l,t)$, $p(l,t)$ in each level come to corresponding limiting values $P^i(l)$, $p(l)$. In the next section the stationary solution $p(l)$ in level l is investigated for different parameters α_0, β_0, c.

III. BEHAVIOR OF THE SYSTEM FOR $\beta_0 = 0$

Let us set $\beta(l) \equiv 0$ in Eqs. (7)–(10). In the limiting case $t \to \infty$ densities of possible configurations satisfy the following relations:

$$P^0(l) = P^0(l)[1-\alpha(l)]^3 + 3P^2(l)[1-\alpha(l)] + P^3(l) \ , \qquad (11)$$

$$P^1(l) = P^0(l)[1-\alpha(l)]^2\alpha(l) + P^1(l)[1-\alpha(l)]^2 + P^2(l)\alpha(l) \ , \qquad (12)$$

$$P^2(l) = P^0(l)\alpha^2(l)[1-\alpha(l)] + 2P^1(l)\alpha(l)[1-\alpha(l)] \ , \qquad (13)$$

$$P^3(l) = P^0(l)\alpha^3(l) + 3P^1(l)\alpha^2(l) \ . \qquad (14)$$

Figure 2 displays the dependence of the density of defects and the appearance intensity on level l. The former exhibits two types of behavior. For $\alpha_0 > \alpha_s$ the density of defects $p(l)$ drops vs level number and comes to a horizontal asymptotic $p(l) = p_s$ for large l. For $\alpha_0 < \alpha_s$ the density $p(l)$ increases to the same asymptotic. For $\alpha_0 = \alpha_s$ the density $p(l)=p_s$ for all levels. The fixed point (α_s, p_s) is stable. For $\alpha_0 \ll \alpha_s$ the density $p(l)$ remains constant (different from p_s) for some first levels [Fig. 2(a)]. This is an unstable fixed point. This value of the unstable fixed point may be easily obtained from Eqs. (11)–(14). Expanding these equations up to the first order of $\alpha(l)$, the expression for the fixed point is as follows:

$$P^0(l) = 2P^1(l), \quad P^2(l) = P^3(l) = 0, \quad p(l) = P^1(l) \ . \qquad (15)$$

The sum of probabilities $P^i(l)$ of all possible configurations of one group is equal to unity:

$$P^0(l) + 3P^1(l) + 3P^2(l) + P^3(l) = 1 \ . \qquad (16)$$

As a consequence the unstable fixed point is

$$P^0(l) = \frac{2}{5}, \quad P^1(l) = \frac{1}{5}, \quad p(l) = \frac{1}{5} \ .$$

It follows also from Eq. (2) that for small $\alpha(l)$ the quantity $\alpha(l+1)$ is also small. Therefore the function $p(l)$ remains close to $1/5$ for some first levels for small appearance intensities $\alpha(l)$.

IV. BEHAVIOR OF THE SYSTEM FOR $\beta_0 \neq 0$, $c \neq 1$

The behavior of the system is more interesting in the case of nontrivial β_0 and c (Fig. 3). For $\alpha_0 < \alpha_{cr}$ the density $p(l)$ monotonically tends to zero for $n \to \infty$, while for $\alpha_0 > \alpha_{cr}$ the density $p(l)$ tends to a limiting value p_s just as in the case without healing of single defects (Fig. 2). It follows from Eq. (4) that the limiting value p_s of density $p(l)$ for $\alpha_0 > \alpha_{cr}$ is equal to the stable fixed point for $\beta_0 = 0$. Consequently, for $l \to \infty$ solutions of Eqs. (2) and (7)–(10) tend to one of two possible limiting point $(0,0)$ and (α_s, p_s), but there is no fixed point (in contrast to Sec. III). The critical value α_{cr} depends on the parameters of healing β_0 and c.

V. BEHAVIOR OF THE SYSTEM FOR $\beta_0 \neq 0, c = 1$

For $c = 1$ a critical value β_{cr} appears, dividing the range of values of the initial healing parameter β_0 into two parts. The relations for $\beta_0 < \beta_{cr}$ (Fig. 4) are similar to the previous case (Figs. 2 and 3). Namely, for $\alpha_0 < \alpha_{cr}$ densities $p(l)$ tend to zero when l increases, while for $\alpha_0 > \alpha_{cr}$ densities $p(l)$ come to the asymptotic $p(l) = \tilde{p}_s$, $\tilde{p}_s < p_s(\beta_0 = 0)$. The value of $\tilde{p}_s$ drops as β_0 increases. Both functions $p(l)$ and $\alpha(l)$ are monotonic (Fig. 4). There are

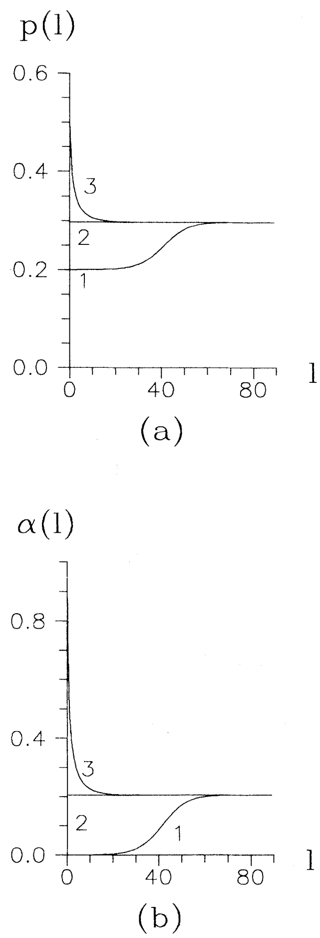

FIG. 2. A relation between level and the corresponding density (a) and appearance intensity (b) of defects. There is no healing of single events ($\beta_0 = 0$). There are three kinds of behavior for different values of initial appearance intensity α_0: (1) $\alpha_0 < \alpha_s$, $\alpha_0 = 0.0001$; (2) $\alpha_0 = \alpha_s \approx 0.205\,91$; (3) $-\alpha_0 > \alpha_s$, $\alpha_0 = 0.99$. For all values of initial appearance intensity α_0 the curves tend to the same constant value when the level grows.

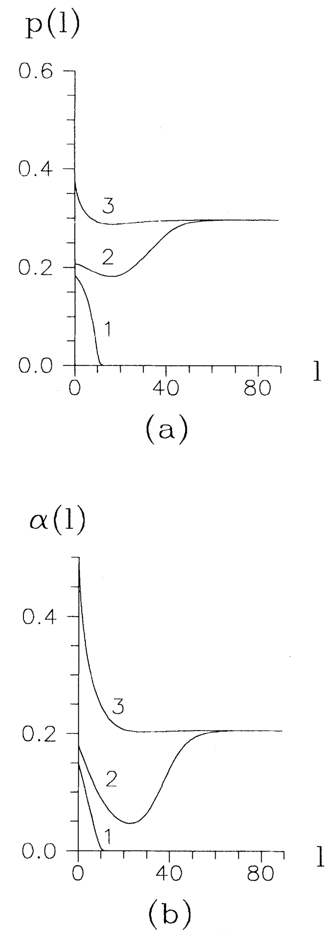

FIG. 3. A relation between level and the corresponding density (a) and appearance intensity (b) of defects. Parameters of healing of single defects are $\beta_0 = 0.5$, $c = 0.9$. There are three kinds of behavior for different values of initial appearance intensity α_0: (1) $-\alpha_0 < \alpha_{cr}$, $\alpha_0 = 0.15$; (2) $\alpha_0 > \alpha_{cr}$, $\alpha_0 = 0.18$; (3) $-\alpha_0 > \alpha_{cr}$, $\alpha_0 = 0.5$. For subcritical values of initial appearance intensity $\alpha_0 < \alpha_{cr}$ the curves of stationary solution tends to 0 when the level grows. For all supercritical values $\alpha_0 > \alpha_{cr}$ the curves tend to a nonzero constant value.

two fixed points. One point ($\alpha_0 = \alpha_s$) is stable, while the other ($\alpha_0 = \alpha_{cr}$) is unstable. As β_0 grows the values of both fixed points come closer and become equal to one another when $\beta_0 = \beta_{cr}$.

For $\beta_0 > \beta_{cr}$ the behavior is quite different. Namely, for arbitrary α_0 both densities of defects $p(l)$ and appearance intensities $\alpha(l)$ tend to zero as $l \to \infty$ (Fig. 5).

VI. THE MAGNITUDE-FREQUENCY LAW AND THE SELF-ORGANIZED CRITICALITY

For the hierarchical model of defects the magnitude-frequency law determines a relation between number and energy of events. It is natural to define an exponential relation between the energy and the level of events:

$$E(l) = 3^l \ . \tag{17}$$

The number of events of a given level is a product of the probability of a new defect appearance and a total number of elements on this level.

$$N(l) = a3^{L-l}[1 - p(l)]\alpha(l) \ . \tag{18}$$

Here a and L are a constant defining the number of events of the highest level and a total number of levels in the system, respectively.

The magnitude-frequency relations of the stationary solution for different values of parameters are shown in Figs. 6(a–d). For $\alpha_0 > \alpha_{cr}$ [Figs. 6(a–c)] the magnitude-frequency law is linear in a log-log plot, with the slope being equal to unity irrespective of the values of the parameters. The linear magnitude-frequency law in the region $\alpha_0 > \alpha_{cr}$ means the self-organized criticality behavior of

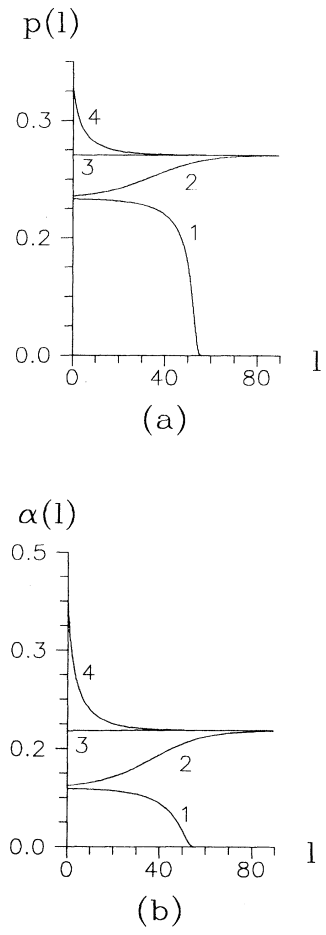

FIG. 4. A relation between level and the corresponding density (a) and appearance intensity (b) of defects. Parameters of healing of single defects are $\beta_0 < \beta_{cr}$, $\beta_0 = 0.1$, $c = 1$. There are four kinds of behavior for different values of initial appearance intensity α_0: (1) $-\alpha_0 < \alpha_{cr}$, $\alpha_0 = 0.095$; (2) $-\alpha_{cr} < \alpha_0 < \alpha_s$, $\alpha_0 = 0.1$; (3) $-\alpha_0 = \alpha_s \approx 0.189\,762$, (4) $\alpha_0 > \alpha_s$, $\alpha_0 = 0.4$. For subcritical values of initial appearance intensity $\alpha_0 < \alpha_{cr}$ the curves of stationary solution tends to 0 when the level grows. For all supercritical values $\alpha_0 > \alpha_{cr}$ the curves tend to a nonzero constant value.

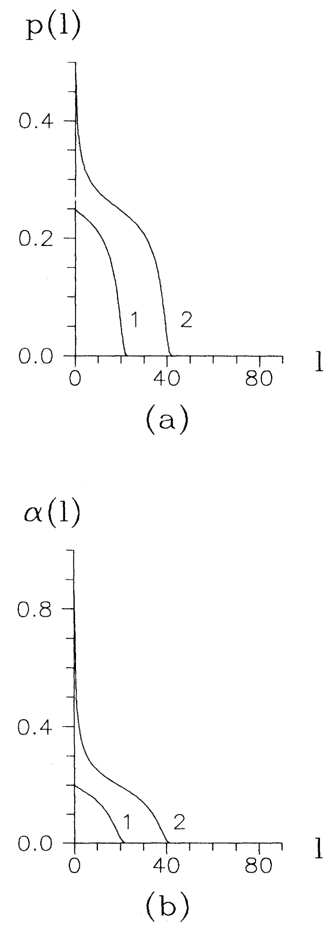

FIG. 5. A relation between level and the corresponding density (a) and appearance intensity (b) of defects. Parameters of healing of single defects are $\beta_0 > \beta_{cr}$, $\beta_0 = 0.3$, $c = 1$. All values of initial appearance intensity α_0 correspond to the same kind of behavior: (1) $-\alpha_0 = 0.2$; (2) $-\alpha_0 = 0.9$. All curves of stationary solution tends to 0 when the level grows.

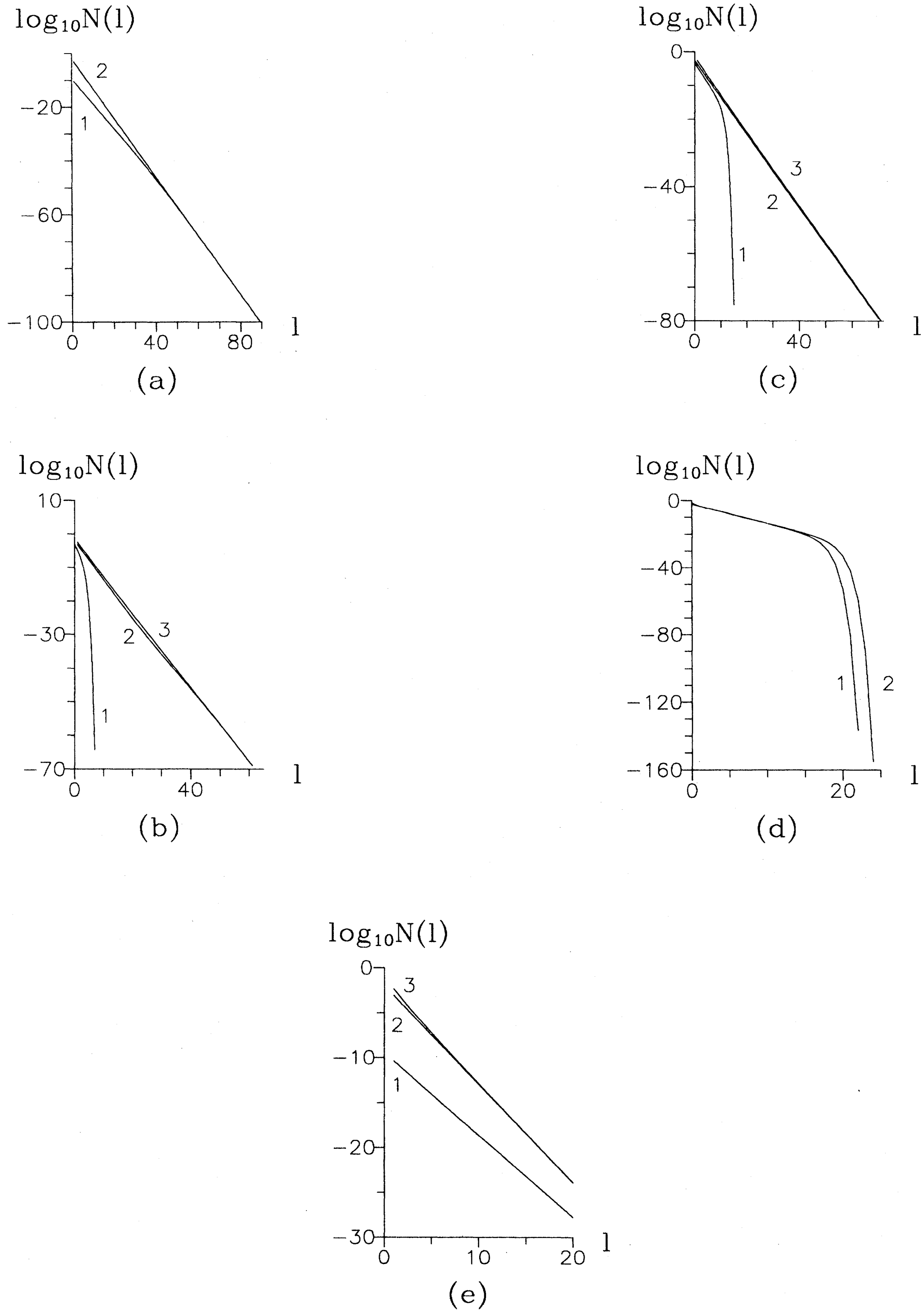

FIG. 6. A magnitude-frequency relation for different values of healing parameters: (a) $-\beta_0 = 0$. All curves have linear behavior for upper levels. A slope of upper levels is equal to unity. (1) $-\alpha_0 = 0.0001$; (2) $\alpha_0 = 0.99$. (b) $-\beta_0 = 0.5$, $c = 0.9$. Subcritical values of initial appearance intensity correspond to an exponential downward bend. All supercritical values correspond to a linear behavior with a slope equal to unity. (1) $-\alpha_0 < \alpha_{cr}$, $\alpha_0 = 0.05$; (2) $\alpha_0 > \alpha_{cr}$, $\alpha_0 = 0.18$; (3) $-\alpha_0 = 0.5$.(c)$-$ $\beta_0 = 0.1$, $c = 1$. Subcritical values of initial appearance intensity correspond to an exponential downward bend. All supercritical values correspond to a linear behavior with a slope equal to unity. (1) $\alpha_0 < \alpha_{cr}$, $\alpha_0 = 0.05$; (2)$\alpha_0 > \alpha_{cr}$, $\alpha_0 = 0.096$; (3) $-\alpha_0 = 0.4$. (d) $-\beta_0 = 0.3$, $c = 1$. All values of initial appearance intensity correspond to an exponential downward bend. (1) $-\alpha_0 = 0.2$; (2) $\alpha_0 = 0.9$. (e)$-$ $\beta_0 = 0$ for 25 initial levels. A slope of the curve corresponding to small values of α_0 is less than 1 for some initial levels. A slope of the curve is more than 1 on a few initial levels for big values of α_0. (1) $-\alpha_0 = 0.0001$; (2) $-\alpha_0 = 0.205\,91$; (3) $-\alpha_0 = 0.99$.

the system. For $\alpha_0 < \alpha_{cr}$ [Figs. 6(b,c)] the considered function $N(l)$ exponentially tends to zero for high values of l. This behavior means the stability, when densities of defects are equal to zero in upper levels of the system. In this case the perturbation appeared on the first level damps on the next levels, and the appearance intensities $\alpha(l)$ tend to zero, so the state of upper levels preserves. For $\alpha_{cr} < \alpha_0 < \alpha_s$ the magnitude-frequency law differs from the straight line but this difference is negligible.

Let us consider the case $\beta_0 = 0$ in detail [Figs. 6(a,e)]. For all values α_0 excluding close vicinity neighborhoods of 0 and 1 the magnitude-frequency law is linear in a log-log plot with a slope equal to unity. For small α_0 the magnitude-frequency law exhibits two parts of straight lines with different slopes and a kind of crossover between them [Fig. 6(a)]. The slope of the second line is equal to unity. The slope of the first interval is less. However, this kind of behavior may be observed only if a sufficiently long interval of energy is considered. When a consideration is restricted to a small interval of energy values the magnitude-frequency law looks as the straight line with a slope less than unity [Fig. 6(e)]. If α_0 tends to 1 then on a few initial levels one can see a linear behavior of the magnitude-frequency law with a slope exceeding unity. Thus it is possible to obtain variations of the slope for a few initial levels, but this variation corresponds to the limiting cases $\alpha_0 \to 0$ and $\alpha_0 \to 1$.

It is possible to change the slope of the magnitude-frequency law by redefining the relation (17) between the energy and level of events. However, the slope also will be independent of the concrete values of parameters α_0, β_0, and c.

VII. CONCLUSIONS

The model considered above is one of a wide class of hierarchical models of defects. It is described by kinetic equations, the number of which grows logarithmically vs the number of degrees of freedom. Just as in the hierarchical models in Refs. [9,10], this system has a stationary solution and a critical point. But in contrast to previous hierarchical models the behavior of the system demonstrates scaling properties not only for the critical point, but for the whole supercritical region. The subcritical behavior of the system corresponds to the stable state of the system, when perturbations appearing on initial levels damp and have no influence to upper levels. The supercritical behavior corresponds to the self-organized criticality and has just the same properties as the Bak s model [3]. Thus, at the critical point the system comes from stability to the self-organized criticality. The self-organized criticality in this model appears for all kinds of healing conditions. Even when there is no healing of single events ($\beta_0 = 0$) a nontrivial stable distribution of defects exists.

The model described in [9] demonstrates a transition from stability to catastrophic behavior. In contrast, the model considered above represents a transition from stability to the self-organized criticality. It shows that the most important kinds of magnitude-frequency relation observed in seismicity and in destruction experiments may be obtained in the simplest hierarchical systems.

An interesting feature of the model is that in the supercritical region the slope of the magnitude-frequency law does not depend on parameters of the model and consequently the former does not reflect the complicated behavior of the system governed by these parameters. The slope equal to unity in the model corresponds to the slope of the Gutenberg-Richter law for the world seismicity. However, it does not reflect the variety in slopes corresponding to different seismic regions. In the model the slope of the magnitude-frequency law in small initial range of sizes of events may differ slightly from the main slope equal to unity. This deviation is not essential for the theoretical investigation that is able to consider arbitrary (large) sizes. Nevertheless this slope of the initial part of the curve (different from unity) may be assumed to be the slope of the magnitude-frequency law in the case of practical simulation, when the range of possible sizes of events is relatively small. To employ this model for practical problems correctly one has to investigate two questions: (i) how high may be the deviation of the slope of the magnitude-frequency law from unity; (ii) how large may be the range of events in which this deviation takes place.

ACKNOWLEDGMENTS

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